Introduction

⇒ Min-max optimization is a problem of interest in several communities including Optimization, Game Theory and Machine Learning.

\[ (\theta^*, \phi^*) = \arg \min_{\theta \in \mathbb{R}^m} \max_{\phi \in \mathbb{R}^n} r(\theta, \phi) \]

⇒ Unfortunately for non-convex non-concave \( r \) little is known, not even when the problem has meaningful solutions.

⇒ Min-max optimization has attracted renewed interest thanks to the application of training GANs.

Bilinear Games vs GANs

⇒ Bilinear games
  \[ \arg \min_{\theta \in \mathbb{R}^m} \max_{\phi \in \mathbb{R}^n} r(\theta, \phi) \]

⇒ Convex Concate
⇒ Players control probabilities directly

⇒ GANs
 \[ \arg \min_{\theta \in \mathbb{R}^m} \max_{\phi \in \mathbb{R}^n} r(\theta, \phi) \]

⇒ Non-Convex Non-Concave
⇒ Players control weights of Generators and Discriminators that compete with each other.

Hidden Bilinear Games

⇒ Hidden Bilinear Game setup

\[ F(\theta) = \left( \begin{array}{c} f(\theta) \\ 1 - f(\theta) \end{array} \right) \]

\[ U = \left( \begin{array}{cc} u_{0,0} & u_{0,1} \\ u_{1,0} & u_{1,1} \end{array} \right) \]

\[ G(\phi) = \left( \begin{array}{c} g(\phi) \\ 1 - g(\phi) \end{array} \right) \]

⇒ Nash equilibrium is \( p, q \) and there is a scalar \( \nu \)

\[ \dot{\theta} = -\nu f(\theta) (g(\phi) - q) \quad \& \quad \dot{\phi} = \nu g(\phi) (f(\theta) - p) \]

Safety of an initialization

⇒ Observe that \( \theta(t) \) cannot traverse stationary points of \( f \).
⇒ This traps \( f(\theta(t)) \) in an interval \( \mu_{\theta(0)} \). Same thing applies to \( g \).
⇒ We call \( (\theta(0), \phi(0)) \) safe if \( \nabla f(\theta(0)) \neq 0 \) and \( \nabla g(\phi(0)) \neq 0 \) and \( p \in \mu_{\theta(0)} \) and \( q \in \mu_{\phi(0)} \)

Possible behaviours of \( f(\theta(t)) \) and \( g(\phi(t)) \)

⇒ Periodic orbits around the \((p, q)\) equilibrium

**Theorem (Periodical Orbits)**

Let \( \theta(0) \) and \( \phi(0) \) be safe initial conditions. Then the orbit \( (\theta(t), \phi(t)) \) is periodic. Additionally

\[ \lim_{t \to \infty} \frac{\int_0^t f(\theta(t)) dt}{t} = p, \quad \lim_{t \to \infty} \frac{\int_0^t g(\phi(t)) dt}{t} = q \]

⇒ Or even convergence to stationary points of \( f \) and \( g \)

**Theorem (Spurious equilibria)**

One can construct functions \( f \) and \( g \) so that for a positive measure set of initial conditions the trajectories converge to fixed points that do not correspond to equilibria of the hidden game.

Gradient-Descent-Ascent

\[ \begin{cases} \theta_{t+1} = \theta_t - \alpha \nabla r(\theta_t, \phi_t) \\ \phi_{t+1} = \phi_t + \alpha \nabla r(\theta_t, \phi_t) \end{cases} \]

Continuous time: CGDA

\[ \begin{cases} \theta_{t+1} = \theta_t - \alpha \nabla r(\theta_t, \phi_t) \\ \phi_{t+1} = \phi_t + \alpha \nabla r(\theta_t, \phi_t) \end{cases} \]

Discrete time: DGDA

Main ideas behind the proof

⇒ Even if \( f, g \) are non-invertible, there exist smooth functions such that \( \theta(t) = X_{\theta(0)}(f(t)), \phi(t) = X_{\phi(0)}(g(t)) \).
⇒ The hidden game of \( f, g \) mimics properties and behaviors of conservative systems. The following quantity is time invariant:

\[ H(f, g) = \int_p \frac{z - p}{\| \nabla f(X_{\theta(0)}(z)) \|^2 + \| \nabla g(X_{\phi(0)}(z)) \|^2} dz \]

⇒ Apply the Poincaré-Bendixon theorem on the planar dynamical system of \((f, g)\) to get periodicity.

Extensions

More than two strategies:

⇒ Let vectors \( F(\theta), G(\phi) \) have size \((N, M)\) whose each coordinate is controlled by disjoint subsets of the vectors \( \theta, \phi \), i.e.

\[ F(\theta) = \left[ f_1(\theta_1) \cdots f_N(\theta_N) \right] \quad G(\phi) = \left[ g_1(\phi_1) \cdots g_M(\phi_M) \right] \]

⇒ CGDA is bounded and diffeomorphic to a volume-preserving system.
⇒ Poincaré recurrence theorem states that volume-preserving bounded systems will, after a sufficiently long but finite time, recurtum to a state very close to their initial state.

Question: Discretization is critical? Answer: No

**Theorem (Spurious equilibria)**

One can choose a learning rate \( \alpha \) and functions \( f, g \) (i.e. sigmoids) for DGDA so that its energy \( H \) is always non-decreasing and for a positive measure set of initial conditions the trajectories converge to fixed points that do not correspond to equilibria of the hidden game.

Full version at arxiv:TBA