Data Science Institute
COLUMBIA UNIVERSITY

SAMPLE-BASED OPTIMAL PRICING

Amine Allouah and Omar Besbes, {ma3411, ob2105}@columbia.edu

1. Introduction

Pricing is central to many industries and academic disciplines ranging from Operations Research to Computer Science and Economics. In the present paper, we study data-driven optimal pricing in low informational environments. We analyze how a decision-maker should price based on a single sample of the willingness-to-pay (WTP) of customers. The decision-maker's objective is to select a general pricing policy with maximum competitive ratio when the WTP distribution is only known to belong to some broad set. We characterize optimal performance across a spectrum of non-parametric families of distributions, λ-regular distributions, two notable special cases being regular and monotone hazard rate distributions. We develop a general approach to obtain parametric lower bounds on the maximin ratio as well parametric upper bounds. The results have implications on the value of samples in absolute terms but also viz., e.g., increased customer competition.

2. Problem formulation

- seller is trying to sell an indivisible good to one buyer
- buyer has a WTP dist'n F
- seller does not know F but observes one sample s from F
- class of pricing mechanisms

- randomizations over arbitrary mappings $\boldsymbol{t}(\cdot)$ from the sample to a posted price. Only assume differentiability of mappings.
- mechanism can be written as $m = (\boldsymbol{\rho}, \boldsymbol{t})$
  ρ_i: weight of mapping $t_i(s)$

- The revenue of the seller using a mechanism m in $\mathcal{M}_N$, if nature is selecting a distribution F,

$$\mathbb{E}_F\left[\sum_{i=1}^{N} \rho_i \; t_i(s) \; 1\{v \geq t_i(s)\}\right].$$

- benchmark $\text{opt}(F) := \sup_{p \geq 0} p \; \overline{F}(p)$.
- performance of mechanism m

$$\inf_{F \in \mathcal{F}} R(m, F) \;=\; \inf_{F \in \mathcal{F}} \frac{\mathbb{E}_F\left[\sum_{i=1}^{N} \rho_i \; t_i(s) \; 1\{v \geq t_i(s)\}\right]}{\text{opt}(F)}.$$

3. Objective

- Class of admissible distributions

$$\mathcal{G} \;=\; \{F : [0, \infty) \to [0, 1] : F \text{ is a cdf and } 0 < \mathbb{E}_F[v] < \infty\}.$$

- **Objective** of the seller: for $\mathcal{F} \subset \mathcal{G}$,

$$\mathcal{R}(\mathcal{M}, \mathcal{F}) \;=\; \sup_{m \in \mathcal{M}} \inf_{F \in \mathcal{F}} R(m, F).$$

4. λ-regular

Definition Fix λ in $[0,1]$. A cdf F is said to be λ-regular on its support $\mathscr{S}_F$ if it admits a density f and if the corresponding λ-virtual value function $\phi_F^\lambda : v \mapsto \lambda v - (1 - F(v))/f(v)$ is non-decreasing over $\mathscr{S}_F$.
Special cases: regular dist'ns $\lambda = 1$ and increasing monotone hazard rate dist'ns $\lambda = 0$
Focus

$$\mathcal{R}(\mathcal{M}, \mathcal{F}_\lambda)$$

5. Contributions

- **unifying framework** to study sample-based optimal pricing
  – connection between reliability theory and pricing
  – systematic approach
- first **impossibility results** (upper bounds) for general (randomized) strategies
- tightest lower bounds to date on sample-based optimal pricing

6. Overview of main results

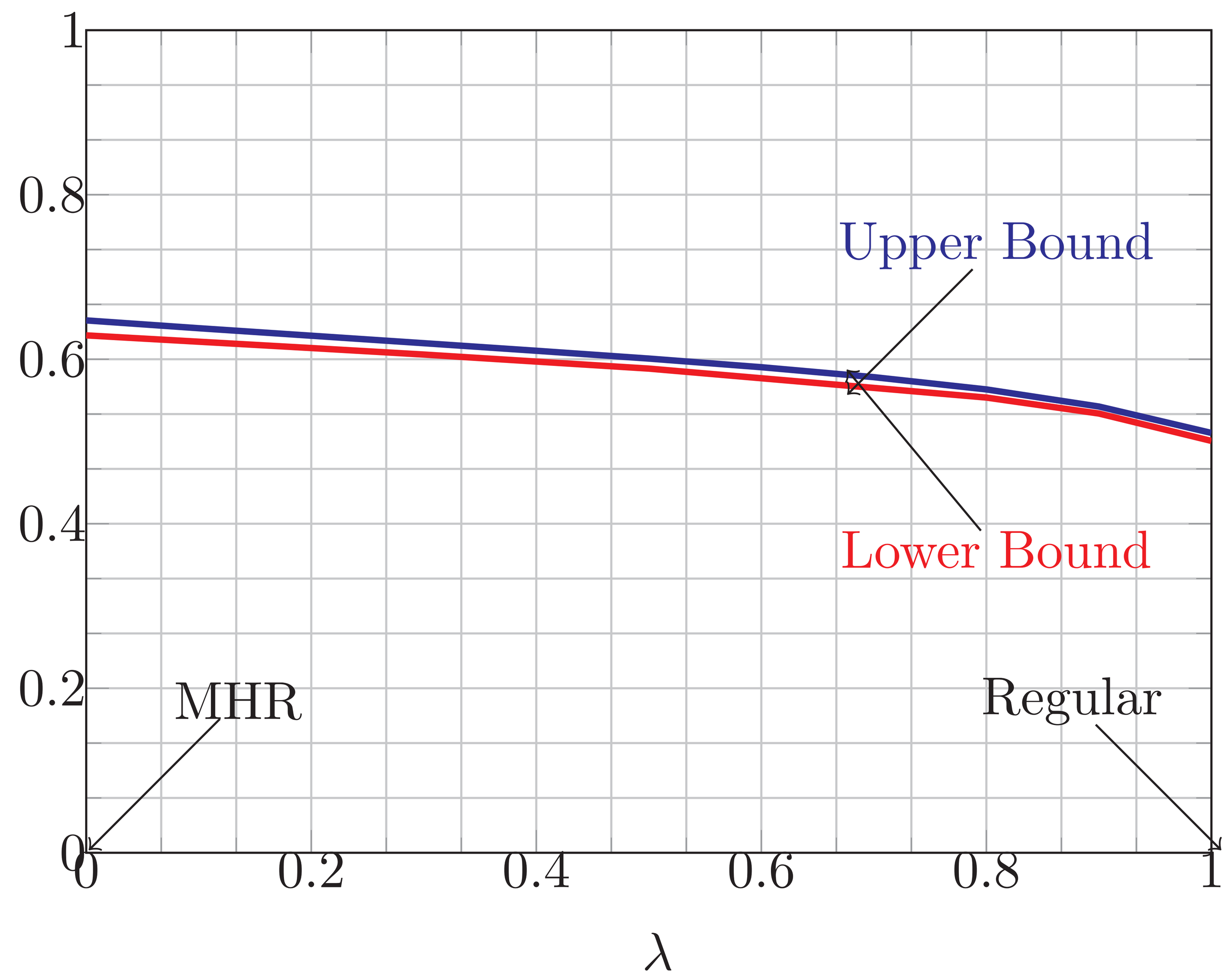

Figure 1: Bounds on $\mathcal{R}(\mathcal{M}, \mathcal{F}_\lambda)$

7. Tractable Bounds

Theorem

$$\mathcal{R}(\mathcal{M}, \mathcal{F}_\lambda) \geq \sup_{N \geq 1} \sup_{\substack{\gamma \in \mathbb{R}_+^N \\ \rho \in \Delta_N}} \inf_{\substack{q_0 \in (0,1], \\ q_{1,i}/q_0 \leq \gamma_i}} \sum_{i=1}^{N} \rho_i \; \underline{\mathcal{C}}_{\lambda,\gamma_i}(q_0, q_{1,i})$$

$$\mathcal{R}(\mathcal{M}, \mathcal{F}_\lambda) \leq \sup_{N \geq 1} \sup_{\substack{\gamma \in \mathbb{R}_+^N \\ \rho \in \Delta_N}} \inf_{\substack{q_0 \in (0,1], \\ q_{1,i}/q_0 \leq \gamma_i}} \sum_{i=1}^{N} \rho_i \; \overline{\mathcal{C}}_{\lambda,\gamma_i}(q_0, q_{1,i})$$

- seller: from infinite dim. to collection of finite dim. spaces!
- nature: from infinite dim. to finite dim. space!
- functionals $\underline{\mathcal{C}}_{\lambda,\gamma}(\cdot,\cdot)$ and $\overline{\mathcal{C}}_{\lambda,\gamma}(\cdot,\cdot)$ driven by appropriate and related dynamic programming recursions

8.1 Approach: Linear Mechanisms

Proposition For any class $\mathcal{F} \subseteq \mathcal{G}$ that is invariant under scaling, it is sufficient to consider randomizations of linear transformations. Namely, we have

$$\mathcal{R}(\mathcal{L}, \mathcal{F}) \;=\; \mathcal{R}(\mathcal{M}, \mathcal{F})$$

where $\mathcal{L} = \{m \in \mathcal{M} : t_i(s) = \gamma_i s\}$

8.2 Approach: Local analysis

Proposition For any mechanism $\gamma > 0$, if $w' \leq r_F/\gamma$,

$$\mathbb{E}_F\left[\gamma s 1\{v \geq \gamma s\} 1\{w \leq s \leq w'\}\right]$$
$$\geq \; \gamma w' \bar{F}(\gamma w) \bar{F}(w) \mathcal{A}_\lambda^L\left(\frac{\bar{F}(w')}{\bar{F}(w)}, \frac{w}{w'}, \frac{\bar{F}(\gamma w')}{\bar{F}(\gamma w)}\right).$$

8.3 Approach: from local to global

Proposition For any $\gamma \in (0,1)$

$$\mathbb{E}_F\left[\gamma s 1\{v \geq \gamma s\} 1\{s \leq r_F/\gamma\}\right] \geq r_F \; \mathcal{D}_{\lambda,\gamma}^L(\bar{F}(r_F), \bar{F}(r_F/\gamma)),$$

where $\mathcal{D}_{\lambda,\gamma}^L(y,x)$ is defined through the dynamic programming recursion

$$\mathcal{D}_{\lambda,\gamma}^L(y,x) = \inf_{z \in \mathcal{B}_\lambda(y,x)} \left\{xz\mathcal{A}_\lambda^L(y/x, \gamma, x/z) + \gamma \mathcal{D}_{\lambda,\gamma}^L(x,z)\right\},$$

for $x, y \in [0,1]^2$ with $\mathcal{B}_\lambda(y,x)$, constraint set parametrized by λ.

9. Conclusion

- developed a general framework for low sample regime that highlights the key drivers of performance
- characterized structure of sample-based optimal pricing and maximin ratio
- additional questions: connecting low sample to large sample regimes...

10. References

Bulow, J. & Klemperer, P. (1996), 'Auctions versus negotiations', *American Economic Review* **86**(1), 180–94.

Fu, H., Immorlica, N., Lucier, B. & Strack, P. (2015), Randomization beats second price as a prior-independent auction, *in* 'Proceedings of the 16th ACM Conference on Economics and Computation', ACM, pp. 323–323.

Huang, Z., Mansour, Y. & Roughgarden, T. (2015), Making the most of your samples, *in* 'Proceedings of the 16th ACM Conference on Economics and Computation', pp. 45–60.